Entangled three-qubit states without concurrence and three-tangle

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Abstract

We provide a complete analysis of mixed three-qubit states composed of a GHZ state and a W state orthogonal to the former. We present optimal decompositions and convex roofs for the three-tangle. Further, we provide an analytical method to decide whether or not an arbitrary rank-2 state of three qubits has vanishing three-tangle. These results highlight intriguing differences compared to the properties of two-qubit mixed states, and may serve as a quantitative reference for future studies of entanglement in multipartite mixed states. By studying the Coffman-Kundu-Wootters inequality we find that, while the amounts of inequivalent entanglement types strictly add up for pure states, this "monogamy" can be lifted for mixed states by virtue of vanishing tangle measures.

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Characterizing and quantifying entanglement of multipartite mixed states is a fundamental issue in quantum information theory, both from a theoretical and a practical point of view [1, 2, 3]. Despite the large number of profound results, e.g. [4, 5, 6, 7, 8, 9, 10, 11, 12, 13], even for the simplest case – the one of three qubits – there is no general solution to this problem. While the entanglement problem is well understood for pure and mixed states of two qubits [1, 2, 3, 14, 15, 16], expanding our knowledge to both higher-dimensional systems and systems of more than two qubits turned out to be a formidable task. Although pure three-qubit states are well characterized by their local invariants and their non-trivial entanglement properties, similarly complete understanding of mixed three-qubit states remains elusive.

In Ref. [15] it has been worked out that mixed states are characterized in terms of the convex roof for pure-state entanglement measures. The seminal paper by Coffman et al. [5] provided a basis for the quantification of three-party entanglement by introducing the so-called residual tangle that led to understanding important subtleties in multipartite quantum correlations [6]. To date, there is no simple analytical method to compute the residual tangle for mixed three-qubit states. Therefore, it may be helpful to study simple cases of such mixed states which, on the one hand, allow for a complete characterization, but on the other hand involve as many features as possible that go beyond the well-known bipartite case.

To this end, we study a particular family of rank-2 mixed three-qubit states. After briefly introducing the basic concepts to describe two-qubit and three-qubit entanglement we first analyze superpositions of GHZ and W states. Then we present the results for the residual tangle of GHZ/W mixtures that lead us to the optimal decompositions, and the corresponding convex roofs. Surprisingly, we obtain the residual tangle not only for mixtures of GHZ and W states, but also for a large part of the Bloch sphere spanned by these states. Finally we discuss the Coffman-Kundu-Wootters (CKW) inequality. Our results provide a rich reference for further work on the quantification of multipartite entanglement.

Important concepts. – The concurrence $C(\phi_{AB})$ measures the degree of bipartite entanglement shared between the parties A and B in a pure two-qubit state $|\phi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. In terms of the coefficients $\{\phi_{00}, \phi_{01}, \phi_{10}, \phi_{11}\}$ of $|\phi_{AB}\rangle$ with respect to an orthonormal basis it is defined as

$$C \;=\; 2|\phi_{00}\phi_{11} - \phi_{01}\phi_{10}| \quad . \tag{1}$$

The concurrence is maximal for Bell states like $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and vanishes for factorized states.

The 3-tangle (or residual tangle) $\tau_3(\psi)$, a measure for three-party entanglement in the three-qubit state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ has been introduced in Ref. [5]. It can be expressed by using the wavefunction coefficients $\{\psi_{000}, \psi_{001}, \ldots, \psi_{111}\}$ as

$$\begin{aligned}
\tau_3 &= 4\ |d_1 - 2d_2 + 4d_3| \qquad (2)\\
d_1 &= \psi_{000}^2\psi_{111}^2 + \psi_{001}^2\psi_{110}^2 + \psi_{010}^2\psi_{101}^2 + \psi_{100}^2\psi_{011}^2\\
d_2 &= \psi_{000}\psi_{111}\psi_{011}\psi_{100} + \psi_{000}\psi_{111}\psi_{101}\psi_{010}\\
&\quad + \psi_{000}\psi_{111}\psi_{110}\psi_{001} + \psi_{011}\psi_{100}\psi_{101}\psi_{010}\\
&\quad + \psi_{011}\psi_{100}\psi_{110}\psi_{001} + \psi_{101}\psi_{010}\psi_{110}\psi_{001}\\
d_3 &= \psi_{000}\psi_{110}\psi_{101}\psi_{011} + \psi_{111}\psi_{001}\psi_{010}\psi_{100} \quad .
\end{aligned}$$

We mention that this definition of the 3-tangle coincides with the modulus of a so-called hyperdeterminant which was introduced by Caley [17, 18]. For the GHZ state

$$|GHZ\rangle \ = \ \frac{1}{\sqrt{2}}\left(|000\rangle \ + \ |111\rangle\right) \qquad (3)$$

the 3-tangle becomes maximal: $\tau_3(GHZ) = 1$, and it vanishes for any factorized state. Remarkably there is a class of entangled three-qubit states for which τ_3 vanishes [6]. This class is represented by the $|W\rangle$ state

$$|W\rangle \ = \ \frac{1}{\sqrt{3}}\left(|100\rangle \ + \ |010\rangle \ + \ |001\rangle\right) \quad . \qquad (4)$$

Now consider, e.g., mixed two-qubit states ω with their decompositions

$$\omega \ = \ \sum_j p_j \pi_j, \qquad \pi_j = \frac{|\phi_j\rangle\langle\phi_j|}{\langle\phi_j|\phi_j\rangle} \quad . \qquad (5)$$

The concurrence of the mixed two-qubit state ω is defined as the average pure-state concurrence minimized over all decompositions

$$C(\omega) \ = \ \min \sum p_j C(\pi_j) \quad . \qquad (6)$$

which is also called a convex-roof extension [3, 15, 19]. The mixed-state 3-tangle is defined analogously. A decomposition that realizes the minimum of the respective function, is called optimal.

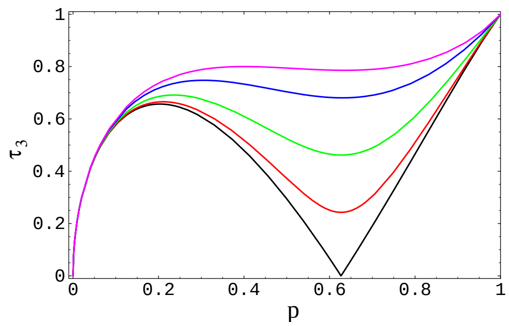

FIG. 1: The 3-tangle of the states $|Z(p,\varphi)\rangle$ in Eq. (8) for various values of $\varphi = \gamma \cdot \frac{2\pi}{3}$ (from top to bottom: $\gamma = 1/2, 1/3, 1/5, 1/10, 0$). The 3-tangle is periodic in φ with a period of $2\pi/3$. Further, $\tau_3(Z)$ vanishes trivially for zero GHZ component, but also for $\varphi = 0$ and non-zero GHZ amplitude $p_0 = \frac{4\sqrt[3]{2}}{3+4\sqrt[3]{2}} \approx 0.627$. This is in striking contrast to the two-qubit case where, in a superposition of a Bell state with an orthogonal unentangled state, the concurrence equals the weight of the Bell state [20], that is the entanglement remains "untouched".

In this paper we study the 3-tangle of rank-2 mixed three-qubit states

$$\rho(p) \ = \ p\, \pi_{\mathrm{GHZ}} \ + \ (1-p)\, \pi_{\mathrm{W}} \tag{7}$$

with $\pi_{\mathrm{GHZ}} = |GHZ\rangle\langle GHZ|$ and $\pi_{\mathrm{W}} = |W\rangle\langle W|$ (notice that $\langle GHZ|W\rangle = 0$).

Superpositions of GHZ and W states. – It is well-known that from the decomposition of a rank-n density matrix $\rho = \sum_{j=1}^{n} p_j\, |j\rangle\langle j|$ into its eigenstates $\{|j\rangle\}$, any other decomposition $\rho = \sum |\chi_l\rangle\langle\chi_l|$ of length $m \geq n$ can be obtained with a unitary $m \times m$ matrix U_{lj} via $|\chi_l\rangle \ = \ \sum_{j=1}^{m} U_{lj}(\sqrt{p_j}\,|j\rangle)$. Hence, the vectors of any decomposition of our states $\rho(p)$ are linear combinations of $|GHZ\rangle$ and $|W\rangle$. Therefore we first analyze the 3-tangle of the states

$$|Z(p,\varphi)\rangle \ = \ \sqrt{p}\,|GHZ\rangle \ - \ e^{i\varphi}\sqrt{1-p}\,|W\rangle \quad . \tag{8}$$

With Eq. (2) we obtain (see also Fig. 1)

$$\tau_3(Z(p,\varphi)) \ = \ \left| p^2 - \frac{8\sqrt{6}}{9}\sqrt{p(1-p)^3}e^{3i\varphi} \right| \quad . \tag{9}$$

We notice that $\tau_3(Z)$ has a non-trivial zero ($\varphi = 0$)

$$p_0 \ = \ \frac{4\sqrt[3]{2}}{3+4\sqrt[3]{2}} \ = \ 0.626851\ldots$$

Mixtures of GHZ and W states. – An immediate consequence of the existence of a finite p_0 with $\tau_3(Z(p_0)) = 0$, together with the permutation symmetry of the states, is that the mixed-state 3-tangle $\tau_3(\rho(p)) = 0$ for $p = p_0$ as well as for all $0 \leq p \leq p_0$. For $p = p_0$ we have the optimal decomposition

$$\begin{aligned} \rho(p_0) &= \frac{1}{3}(\left|Z_0^0\right\rangle\left\langle Z_0^0\right| + \left|Z_0^1\right\rangle\left\langle Z_0^1\right| + \left|Z_0^2\right\rangle\left\langle Z_0^2\right|) \\ \left|Z_0^j\right\rangle &= \sqrt{p_0}\,|GHZ\rangle \;-\; e^{\frac{2\pi i}{3}j}\sqrt{1-p_0}\,|W\rangle \quad . \end{aligned} \tag{10}$$

For $p < p_0$ there is a decomposition with vanishing 3-tangle of the form $\rho(p) = \frac{p}{p_0}\rho(p_0) + \left(1 - \frac{p}{p_0}\right)\pi_{\mathrm{W}}$. For $p > p_0$, all states $\rho(p)$ have non-vanishing 3-tangle.

The decomposition in Eqs. (10) provides a trial decomposition $\mathcal{S}$ for $\rho(p > p_0)$ given by $\mathcal{S} = \{\sqrt{\frac{1}{3}}\,|Z^0\rangle\,, \sqrt{\frac{1}{3}}\,|Z^1\rangle\,, \sqrt{\frac{1}{3}}\,|Z^2\rangle\}$ with $|Z^j\rangle = \left|Z(p, \frac{2\pi}{3}j)\right\rangle$. Its 3-tangle

$$g_{\mathrm{I}}(p) = p^2 - \frac{8\sqrt{6}}{9}\sqrt{p(1-p)^3}\ , \quad p \geq p_0 \tag{11}$$

provides an upper bound for $\tau_3(\rho(p > p_0))$: as the function $g_{\mathrm{I}}(p)$ is not convex in the entire interval $p_0 \leq p \leq 1$, it cannot represent the 3-tangle of $\rho(p)$ for all $p \in [p_0, 1]$.

In order to test this upper bound, we have performed extensive numerical studies (the numerical method can be tested with the two-qubit case as well as with states $\rho(p \leq p_0)$ with vanishing 3-tangle). According to Caratheodory's theorem, for rank-2 states four vectors are sufficient to minimize the 3-tangle. Hence we need to investigate decompositions with 2, 3, or 4 vectors only.

Our numerical results indicate that $\mathcal{S}$ is indeed an optimal decomposition for values of p close to p_0. For larger p, however, we propose the four-vector decomposition $\mathcal{T}$ that consists of three vectors analogous to $\mathcal{S}$ and a *pure* GHZ state:

$$\rho(p) \;= (1-b) \quad |GHZ\rangle\langle GHZ| + \;\frac{b}{3}\;\sum_{j=0}^{2}\left|Z^j(a)\right\rangle\left\langle Z^j(a)\right|$$

where $|Z^j(a)\rangle \equiv \left|Z(a, \frac{2\pi}{3}j)\right\rangle$. The coefficients a and b have to be determined as functions of p. It turns out that the decomposition $\mathcal{S}$ is optimal for $p_0 \leq p \leq p_1 = \frac{1}{2} + \frac{3}{310}\sqrt{465} \equiv 0.70868\ldots$ For $p \in [p_1, 1]$ the decomposition $\mathcal{T}$ has the lowest average 3-tangle, given by the linear function

$$g_{\mathrm{II}}(p) \;=\; 1 \;-\; (1-p)\left(\frac{3}{2} + \frac{1}{18}\sqrt{465}\right) \quad . \tag{12}$$

This is in complete agreement with the numerical results.

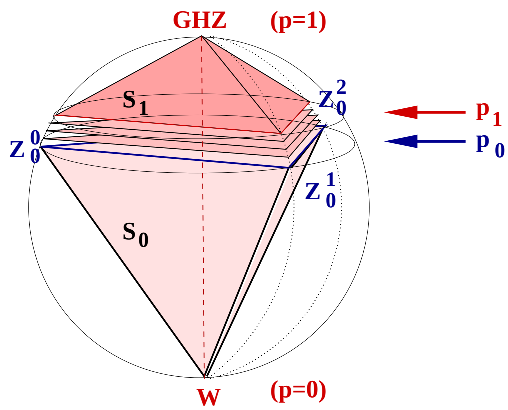

FIG. 2: Bloch sphere for the two-dimensional space spanned by the GHZ state and the W state. The simplex S_0 contains all states with zero 3-tangle. The "leaves" between p_0 and p_1 represent sets of constant 3-tangle, and in the simplex S_1 the 3-tangle is affine (for further explanation, see text).

Convex roofs. – With these findings we can elucidate the affine regions (the convex roofs) of the 3-tangle for any mixture of GHZ and W states. To this end, imagine the two-dimensional space spanned by the GHZ and the W state represented by a Bloch sphere with the GHZ state at its north pole and the W state at the south pole (cf. Fig. 2). All the states $|Z(p,\varphi)\rangle$ are located at the unit sphere, with azimuth φ and p the distance from the south pole along the north-south line (i.e., $p=0$ is the south pole, and $p=1$ is the north pole). The states with zero 3-tangle are represented by the simplex S_0 with corners W, Z_0^0, Z_0^1, Z_0^2.

Now consider planes $P(p)$ parallel to the ground plane of the simplex (i.e., the plane containing the triangle Z_0^0, Z_0^1, Z_0^2) and intersection point p with the north-south line. For $p_0 \leq p \leq p_1$ the 3-tangle is constant in a triangle that has its corners at the intersection points of the plane $P(p)$ and the meridians through Z_0^0, Z_0^1, and Z_0^2 (see Fig. 2).

For $p_1 \leq p \leq 1$ we have another simplex S_1 with affine 3-tangle. The ground plane of this simplex is formed by the last of the "leaves" (at $p=p_1$) described above, and the top corner is the GHZ state. That is, in each plane parallel to the ground plane of this simplex the 3-tangle is constant. The 3-tangle of any point inside S_1 (with distance p' from the south pole, i.e., the GHZ weight equals p') is a convex combination $\alpha \cdot g_{\rm II}(p_1) + \beta \cdot 1$ of $g_{\rm II}(p_1)$ (the 3-tangle in the ground plane of S_1) and of 1, the value for the GHZ state. The coefficients are $\alpha = \frac{1-p'}{1-p_1}$ and $\beta = \frac{p'-p_1}{1-p_1}$. This completes the characterization of the mixed states $\rho(p)$.

CKW inequality. – Given a whole family of mixed three-qubit states with the corresponding 3-tangle one might like to check the CKW relations [5]. To this end, we consider pure three-qubit states $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and introduce the reduced two-qubit density matrices $\rho_{AB} = \mathrm{tr}_C(|\psi\rangle\langle\psi|)$, $\rho_{AC} = \mathrm{tr}_B(|\psi\rangle\langle\psi|)$, and the reduced density matrix of the first qubit $\rho_A = \mathrm{tr}_{BC}(|\psi\rangle\langle\psi|)$. For pure states, one has the "monogamy relation" $4\det(\rho_A) = C(\rho_{AB})^2 + C(\rho_{AC})^2 + \tau_3(\psi)$, that is, the entanglement of qubit A can be either bipartite entanglement with B or C (concurrence), or three-party entanglement with B *and* C (3-tangle). For mixed three-qubit states ρ, Coffman *et al.* found the inequality

$$4\min\left(\det(\rho_A)\right) \ \geq \ C(\rho_{AB})^2 + C(\rho_{AC})^2 \quad . \tag{13}$$

Here, also the 1-tangle has to be minimized for all possible decompositions of ρ. The reduced two-qubit density matrices are defined analogously as $\rho_{AB} = \mathrm{tr}_C(\rho)$, $\rho_{AC} = \mathrm{tr}_B(\rho)$, $\rho_A = \mathrm{tr}_{BC}(\rho)$. We refer to Eq. (13) as the "CKW inequality".

Due to the invariance of $\rho(p)$ with respect to qubit permutations, we can consider the CKW inequality with respect to the first qubit, without loss of generality. It is straightforward to compute the concurrences:

$$C_{AB}^2 + C_{AC}^2 = 2\left(\max\left[0, \frac{2}{3}(1-p) - \sqrt{\frac{p}{3}(2+p)}\right]\right)^2 \quad . \tag{14}$$

This is a monotonously decreasing function for $p \in [0,1]$ that vanishes for $p_C = 7 - \sqrt{45} \approx 0.292$. For the minimum 1-tangle we obtain

$$4\min\ \det\left(\mathrm{tr}_{BC}\rho(p)\right) \ = \ \frac{1}{9}\ (\ 5p^2\ -\ 4p\ +\ 8\ ) \quad . \tag{15}$$

In order to interpret these results, we plot the 1-tangle, the sum of squared concurrences, and the 3-tangle, see Fig. 3. The 1-tangle is always larger than the sum of squared concurrences and 3-tangle. In particular, there is a region $p_C \leq p \leq p_0$ where there is quite substantial 1-tangle, but there is *zero* concurrence and *zero* 3-tangle in the state $\rho(p)$. Our interpretation of this result is that it is not pre-determined whether the entanglement contained in $\rho(p)$ is bipartite entanglement or three-way entanglement. According to the choice of decomposition the entanglement in $\rho(p)$ can be represented either in terms of bipartite correlations, or as three-partite correlations, respectively. This is a curious fact if we recall that the GHZ and the W state are locally inequivalent. Apparently, for mixed states there is no strict monogamy as for pure states. As opposed to this, for "less mixed states" – i.e.,

for $p < p_C$ and for $p > p_0$ – the entanglement stems (mandatorily) to a large extend either from bipartite correlations, or from three-way correlations.

Conclusion. – Summarizing, we have studied the entanglement properties of the family of mixed three-qubit states $\rho(p)$ (cf. Eq. (7)). As opposed to the two-qubit case (where one has a huge variety of optimal decompositions [14, 15]), for our three-qubit states $\rho(p)$ there appears to be only a single type of optimal decomposition (apart from phase rotations). Further, while a two-qubit state of rank n has always an optimal decomposition of length n, this does not hold for the three-qubit states $\rho(p)$. Remarkably, we have obtained the convex roof of the 3-tangle not only for mixtures as in Eq. (7), but for a considerable part of the Bloch sphere (cf. Fig. 2). We mention that, if any of these density matrices is convexly combined with an *arbitrary* three-qubit density matrix, our results provide a non-trivial upper bound for the 3-tangle of the resulting state.

Moreover, we have found that for the entanglement contained in multipartite mixed states there is no strict monogamy, i.e., it can be represented by different types of locally inequivalent quantum correlations.

Finally, we would like to point out that from our results we may draw an important conclusion for arbitrary rank-2 density matrices $\rho = p\,|1\rangle\langle 1| + (1-p)\,|2\rangle\langle 2|$, of three qubits. Also in the general case, there will exist a simplex S_0 which makes it possible to decide on analytical grounds whether or not a rank-2 state has vanishing 3-tangle. This can be seen as follows.

Consider the 3-tangle of the state $|\psi\rangle = |1\rangle + z\,|2\rangle$ with a complex number z (normalization is irrelevant here). The condition for the 3-tangle of this state to be zero (analogously

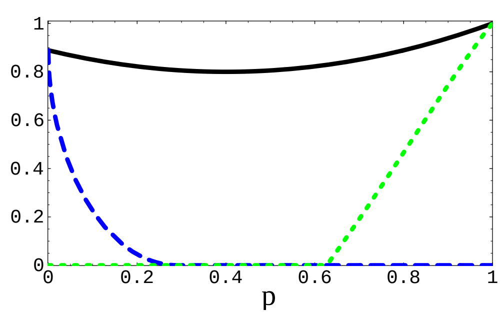

FIG. 3: Plot of the 1-tangle (solid line), the sum of squared concurrences $C(\rho_{AB})^2 + C(\rho_{AC})^2$ (dashed line) and 3-tangle (dotted line) in $\rho(p)$ as a function of p. Clearly, the CKW inequality (13) is satisfied.

to Eq. (9)) is given by a polynomial equation of 4th degree in z. Hence, there are four (in general different) pure states with vanishing 3-tangle that define the simplex S_0 in the corresponding Bloch sphere which contains those density matrices whose 3-tangle is zero. Thus, the 3-tangle of ρ vanishes *iff* it belongs to S_0.

The implications of the concept of the simplex S_0 are reaching even further. It can be generalized for multipartite entanglement monotones of pure states [21, 22, 23]. A method to explicitly write N-qubit entanglement monotones ($N \geq 3$) as certain expectation values of antilinear operators has been presented in Refs. [24, 25]. Let us consider an arbitrary rank-2 N-qubit density matrix. In complete analogy to what was said above, one can find the zeros of such N-tangles on the Bloch sphere which then define a “zero-polyhedron” (rather than a simplex) that contains all the density matrices with vanishing N-tangle.

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